Division Algorithm Theorem

statement :- If $f(x), g(x) \neq 0$ with leading Coefficient in an a unit be two given Polynomials with determinate x over an integral domain R with unity, then there exist unique Polynomials $q(x)$, $r(x)$ in $R[x]$ s.t. $f(x) = g(x) q(x) + r(x)$ where either $r(x) = 0$ or $\deg r(x) < \deg g(x)$.

Proof :- Given $f(x), g(x) \neq 0$ be two Polynomials.

Let $f(x) = a_0 + a_1 x + \dots + a_m x^m$

$g(x) = b_0 + b_1 x + \dots + b_n x^n$

Given b_n is unit

$\exists \; q(x), r(x) \in R[x]$

T.P i) $f(x) = g(x) q(x) + r(x)$

ii) $r(x) = 0$ or $\deg r(x) < \deg g(x)$

$\deg f = m$, $\deg g = n$

Case I If $f(x) = 0$

$f(x) = g(x) \cdot 0 + 0$

ie. $r(x) = 0$, $q(x) = 0$

result is True for $f(x) = 0$

Case II If $f(x) \neq 0$

Sub Case I If $\deg f(x) < \deg g(x)$

$m < n$

then $f(x) = g(x) \cdot 0 + f(x)$

ie $q(x) = 0$, $r(x) = f(x)$

$\therefore$ $\deg r(x) = \deg f(x)$

but we have $\deg f(x) < \deg g(x)$

$\therefore$ $\deg r(x) < \deg g(x)$

result is true for $m < n$

Sub Case II : If $\deg f(x) \geq \deg g(x)$

$m \geq n$

Assumption :- Assume that result is true for all polynomial of degree less than $\deg f(x)$.

Define $f_1(x) = f(x) - a_m b_n^{-1} x^{m-n} g(x)$ — (1)

$f_1(x) = (\cancel{a_0 + a_1}x + \cdots + a_m x^m) - a_m b_n^{-1} x^{m-n}$
$(b_0 + b_1 x + \cdots + b_n x^n)$

Now Coeff. of x^m,

$a_m - a_m b_n^{-1} b_n$

$a_m - a_m = 0$

$0 = 0$

$\therefore$ $\deg f_1(x) < \deg f(x)$

$\deg f_1(x) < m$

By Assumption

$f_1(x) = g(x)\, t(x) + r(x)$ — (2)

$r(x) = 0$ or $\deg r(x) < \deg g(x)$

By ① & ②

$$f(x) - a_m b_n^{-1} x^{m-n} g(x) = g(x) t(x) + r(x)$$

$$f(x) = g(x) [t(x) + a_m b_n^{-1} x^{m-n}] + r(x)$$

$$f(x) = g(x) q(x) + r(x)$$

where $q(x) = t(x) + a_m b_n^{-1} x^{m-n}$

Hence Proved.

<u>Uniqueness</u> :- <u>T.P</u> $q(x)$ and $r(x)$ are unique.

Suppose $f(x) = g(x) q(x) + r(x)$ — ③

$r(x) = 0$ or $\deg r(x) < \deg g(x)$

and $f(x) = g(x) q'(x) + r'(x)$ — ④

$r'(x) = 0$ or $\deg r(x) < \deg g(x)$

By ③ & ④

$$g(x) q(x) + r(x) = g(x) q'(x) + r'(x)$$

$$g(x) [q(x) - q'(x)] = r'(x) - r(x) \quad — ⑤$$

<u>T.P</u> $q(x) - q'(x) = 0$

and $r'(x) - r(x) = 0$

Suppose $q(x) - q'(x) \neq 0$

i.e. $\deg (q(x) - q'(x)) \geq 0$

By ⑤ $\deg [g(x)(q(x) - q'(x))] = \deg [r'(x) - r(x)]$

$$\deg g(x) + \deg [q(x) - q'(x)] = \deg (r'(x) - r(x)) \quad — ⑥$$

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$\because$ $\deg g(x) = n$

L.H.S. of eqn ⑥ $\geq n$

R.H.S also $\geq n$

We know $\deg r(x) < \deg g(x)$

$< n$

and $\deg r'(x) < \deg g(x)$

$< n$

$\therefore$ $\deg r(x) - \deg r'(x) < n$

Contradiction

Hence $q(x) - q'(x) = 0$

$q(x) = q'(x)$

Similarly $r'(x) - r(x) = 0$

$r'(x) = r(x)$

Hence <u>Proved</u>.